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EXISTENCE AND REGULARITY OF SOLUTION FOR A STOCHASTIC CAHN-HILLIARD/ALLEN-CAHN EQUATION WITH UNBOUNDED NOISE DIFFUSION

DIMITRA C. ANTONOPOULOU[†], GEORGIA KARALI[†], AND ANNIE MILLET^{‡*}

ABSTRACT. The Cahn-Hilliard/Allen-Cahn equation with noise is a simplified mean field model of stochastic microscopic dynamics associated with adsorption and desorption-spin flip mechanisms in the context of surface processes. For such an equation we consider a multiplicative space-time white noise with diffusion coefficient of sub-linear growth. Using technics from semi-group theory, we prove existence, and path regularity of stochastic solution depending on that of the initial condition. Our results are also valid for the stochastic Cahn-Hilliard equation with unbounded noise diffusion, for which previous results were established only in the framework of a bounded diffusion coefficient. We prove that the path regularity of stochastic solution depends on that of the initial condition, and are identical to those proved for the stochastic Cahn-Hilliard equation and a bounded noise diffusion coefficient. If the initial condition vanishes, they are strictly less than $2 - \frac{d}{2}$ in space and $\frac{1}{2} - \frac{d}{8}$ in time. As expected from the theory of parabolic operators in the sense of Petrovskiĭ, the bi-Laplacian operator seems to be dominant in the combined model.

Keywords: Stochastic Cahn-Hilliard/Allen-Cahn equation, space-time white noise, convolution semigroup, Galerkin approximations, unbounded diffusion.

1. INTRODUCTION

1.1. The Stochastic equation. We consider the Cahn-Hilliard/Allen-Cahn equation with multiplicative space-time noise:

$$(1.1) \quad \begin{cases} u_t &= -\varrho \Delta (\Delta u - f(u)) + (\Delta u - f(u)) + \sigma(u) \dot{W} & \text{in } \mathcal{D} \times [0, T), \\ u(x, 0) &= u_0(x) & \text{in } \mathcal{D}, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial \mathcal{D} \times [0, T). \end{cases}$$

Here, \mathcal{D} is a rectangular domain in \mathbb{R}^d with $d = 1, 2, 3$, $\varrho > 0$ is a “physical diffusion” constant, f is a polynomial of degree 3 with a positive leading coefficient, such as $f = F'$ where $F(u) = (1 - u^2)^2$ is a double equal-well potential. The “noise diffusion” coefficient $\sigma(\cdot)$ is a Lipschitz function with sub-linear growth, \dot{W} is a space-time white noise in the sense of Walsh [22], and ν is the outward normal vector. In addition, we assume that the initial condition u_0 is sufficiently integrable or regular, depending on the desired results on the solution. Obviously, when $\sigma := 1$, the noise in (1.1) becomes additive.

In this paper, as in [3], we will analyze the more general case of multiplicative noise. Thus, in the sequel we will give sufficient conditions on the initial condition u_0 and on the coefficient σ so that a unique global solution exists with Lipschitz path-regularity. However, unlike [3], we consider a more general Lipschitz coefficient σ with sub-linear growth such that

$$(1.2) \quad |\sigma(u)| \leq C(1 + |u|^\alpha),$$

for some $\alpha \in (0, \frac{1}{9})$ and C a positive constant.

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The stochastic Cahn-Hilliard equation can be considered as a special case of our model. Therefore, our method shows that all the results of [3] on well-posedness and path-regularity for the solution of the stochastic Cahn-Hilliard equation with a multiplicative noise extend to the case where the function σ satisfies the aforementioned sub-linear growth assumption. Furthermore, using the factorization method for the stochastic term we derive a better path regularity than that obtained in [3].

1.2. The physical background. Surface diffusion and adsorption/desorption consist the micro-mechanisms that are typically involved in surface processes or on cluster interface morphology. Chemical vapor deposition, catalysis, and epitaxial growth are surface processes involving transport and chemistry of precursors in a gas phase where the unconsumed reactants and radicals adsorb onto the surface of a substrate so that surface diffusion, or reaction and desorption back to the gas phase is observed. Such processes have been modelled by continuum-type reaction diffusion models where interactions between particles are neglected or treated phenomenologically, [19, 11]. Alternatively, a more precise microscopic description is provided in statistical mechanics theories, [15]. For instance we can consider a combination of Arrhenius adsorption/desorption dynamics, Metropolis surface diffusion and simple unimolecular reaction; the corresponding mesoscopic equation is:

$$(1.3) \quad u_t - D \nabla \cdot \left[\nabla u - \beta u(1-u) \nabla J * u \right] - \left[k_a p(1-u) - k_d u \exp(-\beta J * u) \right] + k_r u = 0.$$

Here, D is the diffusion constant, k_r , k_d and k_a denote respectively the reaction, desorption and adsorption constants while p is the partial pressure of the gaseous species. The partial pressure p is assumed to be a constant, although realistically it is given by the fluids equations in the gas phase. Furthermore, J is the particle-particle interaction energy and β is the inverse temperature.

Stochastic microscopic dynamics such as Glauber and Metropolis dynamics have been analyzed for adsorption/desorption-spin flip mechanisms in the context of surface processes; for more details we refer to the review article [21]. In addition, the Kawasaki and Metropolis stochastic dynamics models describe the diffusion of a particle on a surface, where sites cannot be occupied by more than one particle. Stochastic time-dependent Ginzburg-Landau type equations with additive Gaussian white noise source such as Cahn-Hilliard and Allen-Cahn appear as Model B and Model A respectively in the classical theory of phase transitions according to the universality classification of Hohenberg and Halperin [17]. A simplified mean field mathematical model, associated with the aforementioned mechanisms that describes surface diffusion, particle-particle interactions and as well as adsorption to and desorption from the surface, is a partial differential equation written as a combination of Cahn-Hilliard and Allen-Cahn type equations with noise. The Cahn-Hilliard operator is related to mass conservative phase separation and surface diffusion in the presence of interacting particles. On the other hand, the Allen-Cahn operator is related to adsorption and desorption and serves as a diffuse interface model for antiphase grain boundary coarsening.

At large space-time scales the random fluctuations are suppressed and a deterministic pattern emerges. Such a deterministic model has been analyzed by Katsoulakis and Karali in [20]. The so called mean field partial differential equation has the following form:

$$(1.4) \quad \begin{cases} u_t &= -\varepsilon^2 \varrho \Delta \left(\Delta u - \frac{f(u)}{\varepsilon^2} \right) + \Delta u - \frac{f(u)}{\varepsilon^2}, \\ u(x, 0) &= u_0(x), \end{cases}$$

where $f(u) = F'$ for $F = (1 - u^2)^2/4$ a double-well potential with wells ± 1 , $\varrho > 0$ is the diffusion constant and $0 < \varepsilon \ll 1$ is a small parameter. In [20], the authors rigorously derived the macroscopic cluster evolution laws and transport structure as a motion by mean curvature

depending on surface tension to observe that due to multiple mechanisms an effective mobility speeds up the cluster evolution.

Remark 1.1. *The stochastic equation analyzed in this work is a simplified mean field model for interacting particle systems used in statistical mechanics. These systems are Markov processes set on a lattice corresponding to a solid surface. A typical example is the Ising-type systems defined on a multi-dimensional lattice; see [14]. Assuming that the particle-particle interactions are attractive, then the resulting system's Hamiltonian is nonnegative (attractive potential). Hence, the diffusion constant ϱ of the SPDE (1.1) is considered positive, as in [20].*

Remark 1.2. *Ginzburg-Landau type operators are usually supplemented by Neumann or periodic boundary conditions. In order to obtain an initial and boundary value problem we consider the SPDE (1.1) with the standard homogeneous Neumann boundary conditions on u and Δu . These conditions are frequently used for the deterministic or stochastic Cahn-Hilliard equation; see e.g. [10, 7, 3].*

1.3. Main results. As a first step for a rigorous mathematical analysis of the stochastic model, in Section 2, we will prove existence and uniqueness of a solution to (1.1) when the initial condition u_0 belongs to $L^q(\mathcal{D})$ for $q \in [3, \infty)$ if $d = 1, 2$ and $q \in [6, \infty)$ if $d = 3$. Section 3 describes some possible general assumptions on the domain \mathcal{D} which would lead to the same result and presents the stochastic Cahn-Hilliard equation as a special case of a Cahn-Hilliard/Allen-Cahn stochastic model. Note that the approach used in this paper to solve this non linear SPDE with a polynomial growth is similar to that developed by J.B. Walsh [22] and I. Gyöngy [16] for the stochastic heat equation and related SPDEs. Note that unlike these references, the smoothing effect of the bi-Laplace operator enables us to deal with a stochastic perturbation driven by a space-time white noise in dimension 1 up to 3.

The existence-uniqueness proof is similar to that of Cardon-Weber in [3], and relies on upper estimates of the fundamental solution, Galerkin approximations and the application of a cut-off function. However, the fact that the diffusion coefficient σ is unbounded requires to multiply σ by the cut-off function in order to estimate properly the stochastic integral, and then to use *a priori* estimates for the remaining part.

With our method we prove existence of a global solution under the requirement that σ satisfies the following sub-linear growth condition: $|\sigma(u)| \leq C(1 + |u|^\alpha)$ for some $\alpha \in (0, \frac{1}{9})$ and some positive constant C . Finally note that the upper estimates on the Green's function stated in section 2 obviously show that all the results in [3] can be extended to our framework if σ is bounded. Therefore, one of the main contributions of this paper is to deal with some unbounded noise coefficient σ . Note that we could not apply the technique introduced by S. Cerrai [5] for the stochastic Allen-Cahn equation (see also the work of M. Kuntze and J. van Neerven, [18], for a more general framework) and obtain global solutions for more general diffusion coefficients. This is due to the fact that in our model, in contrast to the Allen-Cahn equation, the Laplace operator is applied to the nonlinearity. However, we believe that global solutions could exist for unbounded noise diffusion coefficients satisfying a growth condition more general than (1.2).

Path regularity of the solution is proved in section 4. If the initial condition vanishes, the domain \mathcal{D} can again be quite general. Otherwise, we have to impose that \mathcal{D} is a rectangle.

Our method shows that all the results of [3] on well-posedness and path regularity of the solution to the stochastic Cahn-Hilliard equation extend to the case of an unbounded noise diffusion. In addition, we prove path regularity of the considered Cahn-Hilliard/Allen Cahn SPDE. The method, based on the factorization method for the deterministic and random forcing terms, yields the same regularity as that proven in [3], where σ is bounded.

As usual we denote by C a generic constant and by $C(s)$ a constant depending on some parameter s . For $p \in [1, \infty]$, the $L^p(\mathcal{D})$ -norm is denoted by $\|\cdot\|_p$. Finally, given real numbers a and b we let $a \vee b$ (resp. $a \wedge b$) denote the maximum (resp. the minimum) of a and b .

2. EXISTENCE OF STOCHASTIC SOLUTION

2.1. Preliminaries. For simplicity and to ease notation, without restriction of generality, we will assume that the “physical diffusion” constant ϱ is equal to 1 and that \mathcal{D} is the unitary cub. Extension to more general domains will be addressed in the next section.

In order to give a mathematical meaning to the stochastic PDE (1.1) we integrate in time and space and use the initial and boundary conditions (see e.g. [22]). For a strict definition of solution, we say that u is a weak (analytic) solution of the equation (1.1) if it satisfies the following weak formulation:

$$(2.1) \quad \begin{aligned} & \int_{\mathcal{D}} \left(u(x, t) - u_0(x) \right) \phi(x) dx = \\ & \int_0^t \int_{\mathcal{D}} \left(-\Delta^2 \phi(x) u(x, s) + \Delta \phi(x) [f(u(x, s)) + u(x, s)] - \phi(x) f(u(x, s)) \right) dx ds \\ & + \int_0^t \int_{\mathcal{D}} \phi(x) \sigma(u(x, s)) W(dx, ds), \end{aligned}$$

for all $\phi \in \mathcal{C}^4(\mathcal{D})$ with $\frac{\partial \phi}{\partial \nu} = \frac{\partial \Delta \phi}{\partial \nu} = 0$ on $\partial \mathcal{D}$. Note that this u stands as a probabilistic “strong solution” since we keep the given space-time white noise and do not only deal with the distribution of the processes.

The random measure $W(dx, ds)$ is the d -dimensional space-time white noise, that is induced by the one-dimensional $(d+1)$ -parameter (with d space variables and one time variable) Wiener process W defined as $W := \{W(x, t) : t \in [0, T], x \in \mathcal{D}\}$. For every $t \geq 0$ we let $\mathcal{F}_t := \sigma(W(x, s) : s \leq t, x \in \mathcal{D})$ denote the filtration generated by W , cf. [22, 3, 1]. Furthermore, we assume that the coefficient $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz function and satisfies the following growth condition for some $\alpha \in (0, 1]$ and $C > 0$:

$$|\sigma(x)| \leq C(1 + |x|^\alpha), \quad \forall x \in \mathbb{R}.$$

2.2. Some results on the Green’s function. Let Δ denote the Laplace operator; we shall use the Green’s function for the operator $\mathcal{T} := -\Delta^2 + \Delta$ on \mathcal{D} with the homogeneous Neumann conditions, that is the fundamental solution to $\partial_t u - \mathcal{T}u = 0$ on \mathcal{D} with the boundary conditions $\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0$ on $\partial \mathcal{D} \times [0, T]$. Let $k = (k_i, i = 1, \dots, d)$ denote a multi-index with non-negative integer components k_i and let $\|k\|^2 := \sum_i k_i^2$. We set $\epsilon_0(x) := \frac{1}{\sqrt{\pi}}$, and for any positive integer j

we define $\epsilon_j(x) := \sqrt{\frac{2}{\pi}} \cos(jx)$. Finally for $k = (k_i) \in \mathbb{N}^d$ and $x \in \mathcal{D}$ let $\epsilon_k(x) := \prod_i \epsilon_i(x_i)$. Then

$(\epsilon_k, k \in \mathbb{N}^d)$ is an orthonormal basis of $L^2(\mathcal{D})$ consisting on eigenfunctions of \mathcal{T} corresponding to the eigenvalues $-\lambda_k^2 - \lambda_k$ where $\lambda_k = \|k\|^2$. Of course, ϵ_0 is related to the null eigenvalue.

Let $S(t) := e^{(-\Delta^2 + \Delta)t}$ be the semi-group generated by the operator \mathcal{T} ; if $u := \sum_k (u, \epsilon_k) \epsilon_k$ then

$$\mathcal{T}u = \sum_k -(\lambda_k^2 + \lambda_k) (u, \epsilon_k)_{L^2(\mathcal{D})} \epsilon_k,$$

and (see e.g. [7, 3]) the convolution semigroup is defined by

$$S(t)U(x) := \sum_k e^{-(\lambda_k^2 + \lambda_k)t} (U, \epsilon_k)_{L^2(\mathcal{D})} \epsilon_k(x),$$

for any U in $L^2(\mathcal{D})$ with the associated Green’s function given by

$$(2.2) \quad G(x, y, t) = \sum_k e^{-(\lambda_k^2 + \lambda_k)t} \epsilon_k(x) \epsilon_k(y)$$

for $t > 0$, $x, y \in \mathcal{D}$. Using the Definition 1.3 of [12], we deduce that $\mathcal{T} = -\Delta^2 + \Delta$ is uniformly strongly parabolic in the sense of Petrovskiĭ. Thus, as proved in [9], the following upper estimates of the Green function G and its various derivatives hold true. Notice that they are similar to those of the Green's function used in [3] for the operator $-\Delta^2$.

Lemma 2.1. *Let G be the Green's function defined by (2.2). Then there exist positive constants c_1 and c_2 such that for any $t \in (0, T]$, any $x, y \in \mathcal{D}$ and any multiindex $k = (k_i, i = 1, \dots, d)$ with $|k| = \sum_{i=1}^d k_i \in \{1, 2\}$, the next inequalities are satisfied:*

$$(2.3) \quad |G(x, y, t)| \leq c_1 t^{-\frac{d}{4}} \exp\left(-c_2 |x - y|^{\frac{4}{3}} t^{-\frac{1}{3}}\right),$$

$$(2.4) \quad |\partial_x^k G(x, y, t - s)| \leq c_1 t^{-\frac{d+|k|}{4}} \exp\left(-c_2 |x - y|^{\frac{4}{3}} t^{-\frac{1}{3}}\right),$$

$$(2.5) \quad |\partial_t G(x, y, t - s)| \leq c_1 t^{-\frac{d+4}{4}} \exp\left(-c_2 |x - y|^{\frac{4}{3}} t^{-\frac{1}{3}}\right).$$

Furthermore, given any $c > 0$ there exists a positive constant $C(c)$ such that

$$(2.6) \quad \int_{\mathbb{R}^d} \exp\left(-c|x|^{\frac{4}{3}} t^{-\frac{1}{3}}\right) dx = C(c) t^{\frac{d}{4}}.$$

For $x \in \mathcal{D}$, $t > s \geq 0$, set

$$(2.7) \quad h(x, t, s) = -c_2 |x|^{\frac{4}{3}} (t - s)^{-\frac{1}{3}}.$$

The following lemma gathers several estimates for integrals of space (respectively time) increments of G . Note once more that the results are the same as those of Lemma 1.8 in [3] and are deduced from the explicit formulation (2.2) of G by using similar arguments.

Lemma 2.2. *Let G be the Green's function defined by (2.2). Given positive constants γ, γ' with $\gamma < (4 - d)$, $\gamma \leq 2$ and $\gamma' < 1 - \frac{d}{4}$, there exists a constant $C > 0$ such that for any $t > s \geq 0$ and any $x, y \in \mathcal{D}$ the next estimates hold true:*

$$(2.8) \quad \int_0^t \int_{\mathcal{D}} |G(x, z, t - r) - G(y, z, t - r)|^2 dz dr \leq C |x - y|^\gamma,$$

$$(2.9) \quad \int_0^s \int_{\mathcal{D}} |G(x, z, t - r) - G(x, z, s - r)|^2 dz dr \leq C |t - s|^{\gamma'},$$

$$(2.10) \quad \int_s^t \int_{\mathcal{D}} |G(x, z, t - r)|^2 dz dr \leq C |t - s|^{\gamma'}.$$

2.3. Integral representation. Using the Green's function, we can present the solution of equation (2.1) in an integral form for any $x \in \mathcal{D}$ and $t \in [0, T]$, that is the following mild solution:

$$(2.11) \quad \begin{aligned} u(x, t) = & \int_{\mathcal{D}} u_0(y) G(x, y, t) dy \\ & + \int_0^t \int_{\mathcal{D}} [\Delta G(x, y, t - s) - G(x, y, t - s)] f(u(y, s)) dy ds \\ & + \int_0^t \int_{\mathcal{D}} G(x, y, t - s) \sigma(u(y, s)) W(dy, ds). \end{aligned}$$

Application of the inequality (2.6) and Hölder's inequality lead to the following bound for the term involving the initial condition.

Lemma 2.3. *Let $G(x, y, t)$ be the Green's function defined by (2.2). For every $1 \leq q < \infty$ and $T > 0$ there exists a constant $C := C(T, q)$ such that*

$$(2.12) \quad \sup_{t \in [0, T]} \|G_t u_0\|_q \leq C \|u_0\|_q,$$

where $G_0 = Id$ and $G_t u_0$ is defined for $t > 0$ by

$$(2.13) \quad G_t u_0(x) := \int_{\mathcal{D}} u_0(y) G(x, y, t) dy.$$

2.4. Truncated equation. In order to prove the existence of the solution u to (2.11), as a first step we consider an appropriated cut-off SPDE, cf. [3]. Let $\chi_n \in C^1(\mathbb{R}, \mathbb{R}^+)$ be a cut-off function satisfying $|\chi_n| \leq 1$, $|\chi'_n| \leq 2$ for any $n > 0$ and

$$\chi_n(x) = \begin{cases} 1 & \text{if } |x| \leq n, \\ 0 & \text{if } |x| \geq n + 1. \end{cases}$$

For fixed $n > 0$, $x \in \mathcal{D}$, $t \in [0, T]$ and $q \in [3, +\infty)$, we consider the following cut-off SPDE:

$$(2.14) \quad \begin{aligned} u_n(x, t) = & \int_{\mathcal{D}} u_0(y) G(x, y, t) dy \\ & + \int_0^t \int_{\mathcal{D}} [\Delta G(x, y, t-s) - G(x, y, t-s)] \chi_n(\|u_n(\cdot, s)\|_q) f(u_n(y, s)) dy ds \\ & + \int_0^t \int_{\mathcal{D}} G(x, y, t-s) \chi_n(\|u_n(\cdot, s)\|_q) \sigma(u_n(y, s)) W(dy, ds). \end{aligned}$$

In this section we suppose that σ satisfies (1.2) with $\alpha \in (0, 1]$, and that the following condition (\mathbf{C}_α) holds:

Condition (\mathbf{C}_α) *One of the following properties (i) or (ii) is satisfied:*

- (i) $d = 1, 2$ and $q \in [3, +\infty)$, or $d = 3$ and $q \in [6, +\infty)$,
- (ii) $d = 3$ and $q \in (3 \vee [6(1 - \alpha)], 6)$.

We show the existence and uniqueness of the solution to the SPDE (2.14) in the set \mathcal{H}_T defined by

$$\mathcal{H}_T := \left\{ u(\cdot, t) \in L^q(\mathcal{D}) \text{ for } t \in [0, T] : u \text{ is } (\mathcal{F}_t)\text{-adapted and } \|u\|_{\mathcal{H}_T} < \infty \right\},$$

where

$$(2.15) \quad \|u\|_{\mathcal{H}_T} := \sup_{t \in [0, T]} E \left(\|u(\cdot, t)\|_q^\beta \right)^{\frac{1}{\beta}},$$

for $\beta \in (\frac{q}{\alpha}, \infty)$ if Condition $(\mathbf{C}_\alpha)(\mathbf{i})$ holds, or for $\beta \in (\frac{q}{\alpha}, \frac{6q}{6-q})$ if Condition $(\mathbf{C}_\alpha)(\mathbf{ii})$ holds.

Remark 2.4. (i) In order to present our results in a more general framework we consider the growth condition (1.2) with $\alpha \in (0, 1]$; the upper bound of α will be restricted in the sequel.

(ii) Note that if $d = 3$, the inequality $6(1 - \alpha) < q < 6$ implies that the interval $(\frac{q}{\alpha}, \frac{6q}{6-q})$ is not empty.

Theorem 2.5. *Let σ be globally Lipschitz and satisfy the assumption (1.2) with $\alpha \in (0, 1]$, let $u_0 \in L^q(\mathcal{D})$ and let Condition (\mathbf{C}_α) hold. Furthermore, let $\beta \in (\frac{q}{\alpha}, +\infty)$ if Condition $(\mathbf{C}_\alpha)(\mathbf{i})$ is satisfied (resp. $\beta \in (\frac{q}{\alpha}, \frac{6q}{6-q})$ if Condition $(\mathbf{C}_\alpha)(\mathbf{ii})$ is satisfied). Then the SPDE (2.14) admits a unique solution u_n in every time interval $[0, T]$ and $u_n \in \mathcal{H}_T$.*

Proof. We define the operators \mathcal{M} and \mathcal{L} on \mathcal{H}_T by

$$(2.16) \quad \mathcal{M}(u)(x, t) := \int_0^t \int_{\mathcal{D}} [\Delta G(x, y, t-s) - G(x, y, t-s)] \chi_n(\|u(\cdot, s)\|_q) f(u(y, s)) dy ds,$$

$$(2.17) \quad \mathcal{L}(u)(x, t) := \int_0^t \int_{\mathcal{D}} G(x, y, t-s) \chi_n(\|u(\cdot, s)\|_q) \sigma(u(y, s)) W(dy, ds),$$

with $u \in \mathcal{H}_T$. Then obviously (2.14) is written as

$$(2.18) \quad u_n(x, t) = \int_{\mathcal{D}} u_0(y) G(x, y, t) dy + \mathcal{M}(u_n)(x, t) + \mathcal{L}(u_n)(x, t).$$

We claim that if $T > 0$ is sufficiently small, then the operator $\mathcal{M} + \mathcal{L}$ is a contraction mapping from \mathcal{H}_T to \mathcal{H}_T .

First we consider the mapping \mathcal{M} . For an arbitrary function $u \in \mathcal{H}_T$, by Minkowski's inequality, (2.3) and (2.4) we have

$$\begin{aligned} \|\mathcal{M}(u)(\cdot, t)\|_q &\leq c_1 \int_0^t (t-s)^{-\frac{d+2}{4}} \\ &\times \left\{ \int_{\mathcal{D}} \left| \int_{\mathcal{D}} \exp\left(-c_2 \frac{|x-y|^{\frac{4}{3}}}{(t-s)^{\frac{1}{3}}}\right) \chi_n(\|u(\cdot, s)\|_q) f(u(y, s)) dy \right|^q dx \right\}^{\frac{1}{q}} ds. \end{aligned}$$

By using Young's inequality with exponents ρ and r in $[1, \infty)$ such that $\frac{1}{\rho} + \frac{1}{r} = \frac{1}{q} + 1$, we obtain for $h(x, t, s) := -c_2 \frac{|x|^{\frac{4}{3}}}{(t-s)^{\frac{1}{3}}}$ defined by (2.7)

$$\begin{aligned} \|\mathcal{M}(u)(\cdot, t)\|_q &\leq c_1 \int_0^t (t-s)^{-\frac{d+2}{4}} \|\exp(h(\cdot, t, s))\|_r \left\| \chi_n(\|u(\cdot, s)\|_q) f(u(\cdot, s)) \right\|_{\rho} ds \\ (2.19) \quad &\leq C \int_0^t (t-s)^{-\frac{d+2}{4} + \frac{d}{4r}} \left\| \chi_n(\|u(\cdot, s)\|_q) f(u(\cdot, s)) \right\|_{\rho} ds, \end{aligned}$$

where the last inequality follows from (2.6). We choose $\rho = \frac{q}{3} \geq 1$ since $q \in [3, \infty)$ and $r \in [1, \infty)$ satisfying $\frac{1}{\rho} + \frac{1}{r} = \frac{1}{q} + 1$. The function f is a polynomial of degree 3, so, for $n \geq 1$ we have

$$(2.20) \quad \left\| \chi_n(\|u(\cdot, s)\|_q) f(u(\cdot, s)) \right\|_{\frac{q}{3}} \leq C n^3.$$

Since $q > d$ we deduce that $-\frac{d+2}{4} + \frac{d}{4r} > -1$; hence the above inequalities yield

$$(2.21) \quad \|\mathcal{M}(u)\|_{\mathcal{H}_T} = \sup_{t \in [0, T]} E \left(\|\mathcal{M}(u(\cdot, t))\|_q^\beta \right)^{\frac{1}{\beta}} \leq C n^3 T^{-\frac{d}{4} + \frac{d}{4r} + \frac{1}{2}}.$$

Therefore, \mathcal{M} is a mapping from \mathcal{H}_T to \mathcal{H}_T . Moreover, for arbitrary u and v in \mathcal{H}_T such that $\|u(\cdot, s)\|_q \leq \|v(\cdot, s)\|_q$, we shall prove that for $q \in [3, \infty)$ and $\rho = \frac{q}{3}$ the next inequality holds true:

$$(2.22) \quad \left\| \chi_n(\|u(\cdot, s)\|_q) f(u(\cdot, s)) - \chi_n(\|v(\cdot, s)\|_q) f(v(\cdot, s)) \right\|_{\rho} \leq C n^3 \|u(\cdot, s) - v(\cdot, s)\|_q.$$

Indeed, we have

$$\begin{aligned} &\left\| \chi_n(\|u(\cdot, s)\|_q) f(u(\cdot, s)) - \chi_n(\|v(\cdot, s)\|_q) f(v(\cdot, s)) \right\|_{\rho} \\ &\leq \left\| [\chi_n(\|u(\cdot, s)\|_q) - \chi_n(\|v(\cdot, s)\|_q)] f(u(\cdot, s)) \right\|_{\rho} + \left\| \chi_n(\|v(\cdot, s)\|_q) [f(u(\cdot, s)) - f(v(\cdot, s))] \right\|_{\rho}. \end{aligned}$$

Note that $\|v(\cdot, s)\|_q \geq \|u(\cdot, s)\|_q$ and

$$\chi_n(\|u(\cdot, s)\|_q) - \chi_n(\|v(\cdot, s)\|_q) = 0 \quad \text{if } \|u(\cdot, s)\|_q \geq n+1.$$

Hence, for $n \geq 1$ and $\rho = \frac{q}{3}$ we obtain the existence of $C > 0$ such that for all $n \geq 1$

$$\begin{aligned} \left\| [\chi_n(\|u(\cdot, s)\|_q) - \chi_n(\|v(\cdot, s)\|_q)] f(u(\cdot, s)) \right\|_{\rho} &\leq C \left(1 + (n+1)^3 \right) \left\| \|u(\cdot, s)\|_q - \|v(\cdot, s)\|_q \right\| \\ &\leq C n^3 \|u(\cdot, s) - v(\cdot, s)\|_q. \end{aligned}$$

Using again the inequality $\|v(\cdot, s)\|_q \geq \|u(\cdot, s)\|_q$, then for any $n \geq 1$ we deduce the existence of $C > 0$ such that

$$\begin{aligned} & \left\| \chi_n(\|v(\cdot, s)\|_q) [f(u(\cdot, s)) - f(v(\cdot, s))] \right\|_\rho \\ & \leq C \chi_n(\|v(\cdot, s)\|_q) \left(1 + \|v(\cdot, s)\|_q^2 + \|u(\cdot, s)\|_q^2 \right) \|u(\cdot, s) - v(\cdot, s)\|_q \\ & \leq C n^2 \|u(\cdot, s) - v(\cdot, s)\|_q. \end{aligned}$$

holds for any $n \geq 1$. Thus, (2.22) holds true.

Inequality (2.22) and an argument similar to that used for proving (2.19) yield

$$\begin{aligned} & \|\mathcal{M}(u)(\cdot, t) - \mathcal{M}(v)(\cdot, t)\|_q \\ & \leq \int_0^t |t - s|^{-\frac{d+2}{4} + \frac{d}{4r}} \left\| \chi_n(\|u(\cdot, s)\|_q) f(u(\cdot, s)) - \chi_n(\|v(\cdot, s)\|_q) f(v(\cdot, s)) \right\|_\rho ds \\ (2.23) \quad & \leq C n^3 \int_0^t |t - s|^{-\frac{d+2}{4} + \frac{d}{4r}} \|u(\cdot, s) - v(\cdot, s)\|_q ds. \end{aligned}$$

Therefore, by inequality (2.23) and Hölder's inequality, since $\beta \in [q, \infty)$, we deduce

$$\begin{aligned} \|\mathcal{M}(u) - \mathcal{M}(v)\|_{\mathcal{H}_T} & \leq C n^3 \sup_{t \in [0, T]} \left\{ E \left| \int_0^t |t - s|^{-\frac{d+2}{4} + \frac{d}{4r}} \|u(\cdot, s) - v(\cdot, s)\|_q ds \right|^\beta \right\}^{1/\beta} \\ & \leq C n^3 T^{(-\frac{d+2}{4} + \frac{d}{4r} + 1)} \sup_{t \in [0, T]} E \left(\|u(\cdot, t) - v(\cdot, t)\|_q \right) \\ (2.24) \quad & \leq C n^3 T^{-\frac{d+2}{4} + \frac{d}{4r} + 1} \|u - v\|_{\mathcal{H}_T}. \end{aligned}$$

Obviously, by (2.21) and (2.24) it follows that for fixed $n \geq 1$ and $T > 0$, the map \mathcal{M} is Lipschitz from \mathcal{H}_T to \mathcal{H}_T .

For the mapping \mathcal{L} defined in terms of a stochastic integral, at first notice that since $\alpha \in (0, 1]$, the inequality $\beta > \frac{q}{\alpha}$ yields $\beta \in (q, \infty)$. Thus the Hölder, Burkholder and Minkowski inequalities, and the growth condition (1.2) on σ yield

$$\begin{aligned} E \|\mathcal{L}(u(\cdot, t))\|_q^\beta & \leq C \int_{\mathcal{D}} E |\mathcal{L}(u(x, t))|^\beta dx \\ & \leq C \int_{\mathcal{D}} E \left| \int_0^t \int_{\mathcal{D}} |G(x, y, t - s) \chi_n(\|u(\cdot, s)\|_q) \sigma(u(y, s))|^2 dy ds \right|^{\beta/2} dx \\ & \leq C \left(E \int_0^t \left\| \int_{\mathcal{D}} G^2(\cdot, y, t - s) \chi_n(\|u(\cdot, s)\|_q) [1 + |u(y, s)|^{2\alpha}] dy \right\|_{\beta/2} ds \right)^{\beta/2}. \end{aligned}$$

Since $\beta \in (\frac{q}{\alpha}, \infty)$, we have $\frac{2\alpha}{q} > \frac{2}{\beta}$ and we may choose $\bar{r} \in (1, \infty)$ such that $\frac{2\alpha}{q} + \frac{1}{\bar{r}} = \frac{2}{\beta} + 1$. Let once more $h(x, t, s)$ be defined by (2.7); Young's inequality and (2.3) imply

$$\begin{aligned} E \|\mathcal{L}(u(\cdot, t))\|_q^\beta & \leq C \left(E \int_0^t (t - s)^{-\frac{d}{2}} \left\| \exp(h(\cdot, t, s)) \right\|_{\bar{r}} \chi_n(\|u(\cdot, s)\|_q) \left\| [1 + |u(\cdot, s)|^{2\alpha}] \right\|_{\frac{q}{2\alpha}} ds \right)^{\beta/2} \\ & \leq C \left(E \int_0^t (t - s)^{-\frac{d}{2} + \frac{d}{4\bar{r}}} (1 + n^{2\alpha}) ds \right)^{\beta/2}. \end{aligned}$$

Note that the inequalities $d < 4$, $q \geq 3$, $\alpha \in (0, 1]$ and $\beta > \frac{q}{\alpha}$ yield $-\frac{d}{2} + \frac{d}{4\bar{r}} > -1$. Hence, for any $u \in \mathcal{H}_T$ we obtain the existence of $C > 0$ such that

$$(2.25) \quad \|\mathcal{L}(u)\|_{\mathcal{H}_T} \leq C(1 + n^\alpha) T^{\frac{1}{2}[-\frac{d}{2} + \frac{d}{4\bar{r}} + 1]}$$

holds for every $n \geq 1$, and therefore, \mathcal{L} is also a mapping from \mathcal{H}_T to \mathcal{H}_T . Recall that σ is Lipschitz. Therefore, an argument similar to that used to prove (2.22) with q instead of ρ shows

that for $u, v \in \mathcal{H}_T$, we have

$$(2.26) \quad \|\delta(u, v, \cdot, s)\|_q \leq C(1 + n^\alpha) \|u(\cdot, s) - v(\cdot, s)\|_q,$$

for

$$\delta(u, v, y, s) := \chi_n(\|u(\cdot, s)\|_q) \sigma(u(y, s)) - \chi_n(\|v(\cdot, s)\|_q) \sigma(v(y, s)).$$

Recall that $\alpha \in (0, 1]$ and $\beta > \frac{q}{\alpha}$, so that $\beta > q$; thus the Hölder, Burkholder-Davies-Gundy and Minkowski inequalities together with (2.3) yield for u, v in \mathcal{H}_T

$$\begin{aligned} E \|\mathcal{L}(u)(\cdot, s) - \mathcal{L}(v)(\cdot, s)\|_q^\beta &\leq C \int_{\mathcal{D}} E |\mathcal{L}(u)(\cdot, s) - \mathcal{L}(v)(\cdot, s)|^\beta dx \\ &\leq C \int_{\mathcal{D}} \left| \int_0^t \int_{\mathcal{D}} G^2(x, y, t-s) |\delta(u, v, y, s)|^2 dy ds \right|^{\beta/2} dx \\ &\leq CE \left| \int_0^t (t-s)^{-\frac{d}{2}} \left\| \exp(h(\cdot, t, s)) * \delta^2(u, v, \cdot, s) \right\|_{\beta/2} ds \right|^{\beta/2}. \end{aligned}$$

The inequality $\beta > q$ implies the existence of $r_2 \in (1, \infty)$ such that $\frac{2}{\beta} + 1 = \frac{2}{q} + \frac{1}{r_2}$. Using once more the assumptions on q, α and β in Condition (C_α) , in particular the assumption $\beta(6-q) < 6q$ for $d = 3$ and $q \in [3, 6)$, we deduce $-\frac{d}{2} + \frac{d}{4r_2} > -1$. Thus Young's inequality and (2.26) imply

$$\begin{aligned} E \|\mathcal{L}(u)(\cdot, s) - \mathcal{L}(v)(\cdot, s)\|_q^\beta &\leq CE \left| \int_0^t (t-s)^{-\frac{d}{2} + \frac{d}{4r_2}} \|\delta(u, v, \cdot, s)\|_{q/2} ds \right|^{\beta/2} \\ &\leq C(1 + n^{\alpha\beta}) T^{(-\frac{d}{2} + \frac{d}{4r_2} + 1)\frac{\beta}{2}} \sup \left\{ E \|u(\cdot, s) - v(\cdot, s)\|_q^\beta : t \in [0, T] \right\}, \end{aligned}$$

and therefore,

$$(2.27) \quad \|\mathcal{L}(u) - \mathcal{L}(v)\|_{\mathcal{H}_T} \leq C(1 + n^\alpha) T^{-\frac{d}{4} + \frac{d}{8r_2} + \frac{1}{2}} \|u - v\|_{\mathcal{H}_T}.$$

So, for fixed n and $T > 0$, the map \mathcal{L} is also a Lipschitz mapping from \mathcal{H}_T to \mathcal{H}_T .

The upper estimates (2.24) and (2.27) imply that the mapping $\mathcal{M} + \mathcal{L}$ is Lipschitz from \mathcal{H}_T to \mathcal{H}_T with the Lipschitz constant bounded by

$$C(n, T) := C \left[n^3 T^{-\frac{d+2}{4} + \frac{d}{4r} + 1} + C n^\alpha T^{-\frac{d}{4} + \frac{d}{8r_2} + \frac{1}{2}} \right].$$

For fixed $n \geq 1$, there exists $T_0(n)$ sufficiently small (which does not depend on u_0) such that $C(n, T) < 1$ for $T \leq T_0(n)$, so that $\mathcal{M} + \mathcal{L}$ is a contraction mapping from the space \mathcal{H}_T into itself. Thus for $T \leq T_0(n)$, the map $\mathcal{M} + \mathcal{L}$ has a unique fixed point in the set $\{u \in \mathcal{H}_T : u(\cdot, 0) = u_0\}$. This implies that in $[0, T]$, for $T \leq T_0(n)$, there exists a unique solution u_n for the SPDE (2.14).

If $T > T_0(n)$, let $\bar{u}_0(x) = u_n(x, T_0(n))$ and $\bar{W}(t, x) = W(T_0(n) + t, x)$; then \bar{W} is a space-time white noise related to the filtration $(\mathcal{F}_{T_0(n)+t}, t \geq 0)$ independent of $\mathcal{F}_{T_0(n)}$. A similar argument proves the existence and uniqueness of the solution \bar{u}_n to an equation similar to (2.14) with u_0 and W replaced by \bar{u}_0 and \bar{W} respectively. Hence, (2.14) has a unique solution u_n on the interval $[0, 2T_0(n)]$, defined by $u_n(x, t) := \bar{u}_n(x, t - T_0(n))$ for $t \in [T_0(n), 2T_0(n)]$. Since there exists $N \geq 1$ such that $NT_0(n) \geq T$ an easy induction argument concludes the proof. \square

2.5. Some bound for the stochastic integral. We shall prove moment estimates for the (space-time) uniform norm for $\mathcal{L}(u_n)$ which will be needed later.

We set

$$\|\mathcal{L}(u_n)\|_{L^\infty} := \sup_{t \in [0, T]} \sup_{x \in \mathcal{D}} |\mathcal{L}(u_n)(x, t)|.$$

Lemma 2.6. *Let σ satisfy Condition (1.2) with $\alpha \in (0, 1]$, let Condition (\mathbf{C}_α) hold, and let u_n be the solution to the SPDE (2.14). Furthermore, suppose that $q > \frac{2\alpha d}{4-d}$. Then for any $p \in [1, \infty)$ there exists a positive constant $C_p(T)$ such that for every $n \geq 1$, we have:*

$$(2.28) \quad E\left(\|\mathcal{L}(u_n)\|_{L^\infty}^{2p}\right) \leq C_p(T)n^{2\alpha p}.$$

Proof. Since $d < 4$, $\alpha \in (0, 1]$ and $q \in [3, \infty)$, we have $q > 2\alpha$ and may choose $\tilde{q} \leq q$ with $\tilde{q} > (2\alpha) \vee \frac{2\alpha d}{4-d}$. For $t \in [0, T]$, using the Burkholder-Davis-Gundy inequality, (2.3), the growth condition (1.2) on σ and Hölder's inequality with conjugate exponents $\frac{\tilde{q}}{2\alpha}$ and $\frac{\tilde{q}}{\tilde{q}-2\alpha}$, we obtain for any $t \in [0, T]$ and $x \in \mathcal{D}$

$$\begin{aligned} E|\mathcal{L}(u)(x, t)|^{2p} &\leq C_p E\left|\int_0^t \int_{\mathcal{D}} (t-s)^{-\frac{d}{2}} \exp(h(\cdot, t, s)) \chi_n(\|u_n(\cdot, s)\|_q) [1 + |u_n(y, s)|^{2\alpha}] dy ds\right|^p \\ &\leq C_p E\left|\int_0^t \int_{\mathcal{D}} (t-s)^{-\frac{d}{2}} \left\|\exp(h(\cdot, t, s))\right\|_{\frac{\tilde{q}}{\tilde{q}-2\alpha}} \chi_n(\|u_n(\cdot, s)\|_q) [1 + \|u_n(y, s)\|_q^{2\alpha}] ds\right|^p \\ &\leq C_p \left|\int_0^t (t-s)^{-\frac{d}{2} + \frac{d(\tilde{q}-2\alpha)}{4\tilde{q}}} n^{2\alpha} ds\right|^p \leq C_p n^{2\alpha p}, \end{aligned}$$

where as above we let $h(x, t, s)$ be defined by (1.2). The last inequality holds provided that $-\frac{d}{4}\left(1 + \frac{2\alpha}{\tilde{q}}\right) > -1$ which holds true since $q \geq \tilde{q} > \frac{2\alpha d}{4-d} \vee (2\alpha)$.

Similar computations using (2.3), (2.4) and the Taylor formula imply that for $x, \xi \in \mathcal{D}$ and $t \in [0, T]$, we have for $\lambda \in (0, 1)$, $\tilde{q} \leq q$, $p \in [1, \infty)$ and $n \geq 1$:

$$\begin{aligned} E|\mathcal{L}(u)(x, t) - \mathcal{L}(u)(\xi, t)|^{2p} &\leq C_p E\left|\int_0^t \int_{\mathcal{D}} |G(x, y, t-s) - G(\xi, y, t-s)|^2 \chi_n(\|u_n(\cdot, s)\|_q) \right. \\ (2.29) \quad &\quad \times [1 + |u_n(y, s)|^{2\alpha}] dy ds\left|^p \\ &\leq C_p |x - \xi|^{2\lambda p} E\left|\int_0^t (t-s)^{-\frac{(d+1)\lambda}{2}} (t-s)^{-\frac{d}{2}(1-\lambda)} \chi_n(\|u_n(\cdot, s)\|_q) \right. \\ &\quad \times \left\|\exp(h(\cdot, t, s))\right\|_{\frac{\tilde{q}}{\tilde{q}-2\alpha}} [1 + \|u_n(\cdot, s)\|_{\tilde{q}}^{2\alpha}] ds\left|^p \\ &\leq C_p |x - \xi|^{2\lambda p} n^{2\alpha p} \left|\int_0^t (t-s)^{-\frac{d+\lambda}{2} + \frac{d(\tilde{q}-2\alpha)}{4\tilde{q}}} ds\right|^p \\ (2.30) \quad &\leq C_p(T) n^{2\alpha p} |x - \xi|^{2\lambda p}, \end{aligned}$$

provided that $-\frac{d+\lambda}{2} + \frac{d(\tilde{q}-2\alpha)}{4\tilde{q}} > -1$, which holds true if $0 \leq \lambda < \left(2 - \frac{d}{2}\right) \wedge 1$ and $\tilde{q} > \frac{2\alpha d}{4-d-2\lambda}$.

Hence, for $q > \frac{2\alpha d}{4-d}$ one can find $\lambda \in (0, 1)$ small enough and \tilde{q} as above.

Using again the Taylor formula, (2.3) and (2.5), we obtain, for $0 \leq t' \leq t \leq T$ and $\mu \in [0, 1]$

$$\begin{aligned} E|\mathcal{L}(u)(x, t) - \mathcal{L}(u)(x, t')|^{2p} &\leq C_p E\left|\int_0^t \int_{\mathcal{D}} [|G(x, y, t-s)|^{2(1-\mu)} + |G(x, y, t'-s)|^{2(1-\mu)}] \right. \\ &\quad \times |G(x, y, t-s) - G(x, y, t')|^{2\mu} \chi_n(\|u_n(\cdot, s)\|_q) [1 + |u_n(y, s)|^{2\alpha}] dy ds\left|^p \\ &\leq |t - t'|^{2\mu p} E\left|\int_0^t (t-s)^{-2\mu(\frac{d}{4}+1)-(1-\mu)\frac{d}{2}} \left\|\exp(h(\cdot, t, s))\right\|_{\frac{\tilde{q}}{\tilde{q}-2\alpha}} \right. \\ &\quad \times (1 + \|u_n(\cdot, s)\|_{\tilde{q}}^{2\alpha}) \chi_n(\|u_n(\cdot, s)\|_q) ds\left|^p \\ (2.31) \quad &\leq C_p(T) |t - t'|^{2\mu p} n^{2\alpha p}, \end{aligned}$$

where the last inequality holds true if $-\frac{d}{2} - 2\mu + \frac{d}{4}\left(1 - \frac{2\alpha}{q}\right) > -1$; this is similar to the previous requirement used to prove (2.30) replacing $\frac{\lambda}{2}$ by 2μ . Thus, since $q > \frac{2\alpha d}{4-d}$, we may find $\tilde{q} \in (\frac{2\alpha d}{4-d}, q]$ and $\mu \in (0, 1)$ which satisfy this constraint, and such that (2.31) holds for any $p \in [1, +\infty)$.

The upper estimates (2.30), (2.31) imply the existence of some positive constants λ and μ , and given $p \in [1, \infty)$ of some positive constant $C_p(T)$ (independent of n) such that for $x, x' \in \mathcal{D}$ and $t, t' \in [0, T]$, we have for every $n \geq 1$

$$E|\mathcal{L}(u)(x, t) - \mathcal{L}(u)(x', t')|^{2p} \leq C_p(T)[|x - \xi|^{2\lambda p} + |t - t'|^{2\mu p}]n^{2\alpha p}.$$

Therefore, the Garsia-Rodemich-Rumsey Lemma yields the upper estimate (2.28). \square

2.6. Galerkin approximation. In this section we need some stronger integrability condition on the initial condition u_0 , which is required to be in $L^4(\mathcal{D})$. More precisely, we suppose that the following condition $(\tilde{\mathbf{C}}_\alpha)$ is satisfied.

Condition $(\tilde{\mathbf{C}}_\alpha)$ *One of the following properties is satisfied:*

- (i) *Either $d = 1, 2$ and $q \in [4, \infty)$, or $d = 3$ and $q \in [6, \infty)$;*
- (ii) *$d = 3$ and $q \geq 4$ is such that $q \in (6(1 - \alpha) \vee (6\alpha), 6)$.*

Note that if Condition $(\tilde{\mathbf{C}}_\alpha)$ is satisfied, we have $\|\cdot\|_4 \leq C\|\cdot\|_q$ for some positive constant C .

For any $n \geq 1$, we define

$$v_n := u_n - \mathcal{L}(u_n).$$

Then, formally, v_n satisfies the following equation:

$$(2.32) \quad \begin{aligned} \partial_t v_n + [\Delta^2 - \Delta]v_n - (\Delta - Id)\left(\chi_n(\|v_n + \mathcal{L}(u_n)\|_q)f(v_n + \mathcal{L}(u_n))\right) &= 0 \text{ in } \mathcal{D} \times [0, T], \\ v_n(x, 0) &= u_0(x) \text{ in } \mathcal{D}, \\ \frac{\partial v_n}{\partial \nu} &= \frac{\partial \Delta v_n}{\partial \nu} = 0 \text{ on } \partial\mathcal{D} \times [0, T]. \end{aligned}$$

For a strict definition of solution, we say that v_n is a weak solution of the above equation (2.32) if for all $\phi \in \mathcal{C}^4(\mathcal{D})$ with $\frac{\partial \phi}{\partial \nu} = \frac{\partial \Delta \phi}{\partial \nu} = 0$ on $\partial\mathcal{D}$, we have:

$$\begin{aligned} \int_{\mathcal{D}} (v_n(x, t) - u_0(x))\phi(x) dx &= \int_0^t \int_{\mathcal{D}} \left\{ [-\Delta^2 + \Delta]\phi(x)v_n(x, s) \right. \\ &\quad \left. + [\Delta\phi(x) - \phi(x)]\chi_n(\|v_n + \mathcal{L}(u_n)\|_q)f(v_n + \mathcal{L}(u_n)) \right\} dx ds. \end{aligned}$$

Using the Green's function G defined by (2.2), we deduce the integral form of this equation:

$$(2.33) \quad \begin{aligned} v_n(x, t) &= \int_{\mathcal{D}} u_0(y)G(x, y, t) dy + \int_0^t \int_{\mathcal{D}} [\Delta G(x, y, t-s) - G(x, y, t-s)] \\ &\quad \times \chi_n(\|v_n + \mathcal{L}(u_n)\|_q)f(v_n(y, s) + \mathcal{L}(u_n)(y, s)) dy ds. \end{aligned}$$

We will use the Galerkin method to prove the existence of the solution v_n for the equation (2.32). Let us denote by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ the eigenvalues of Neumann Laplacian operator inducing $\{w_i\}_{i=0}^\infty$ as an orthonormal basis of $L^2(\mathcal{D})$ of eigenfunctions, i.e., $(w_i, w_j)_{L^2(\mathcal{D})} = \delta_{ij}$ and

$$(2.34) \quad -\lambda_i w_i = \Delta w_i \text{ in } \mathcal{D}, \quad \frac{\partial w_i}{\partial \nu} = 0 \text{ on } \partial\mathcal{D} \text{ for } i = 0, 1, 2, \dots$$

Let P_m denote the orthogonal projection from $L^2(\mathcal{D})$ onto $\text{span}\{w_0, w_1, \dots, w_m\}$. For every $m = 0, 1, 2, \dots$ we consider the function v_n^m

$$v_n^m(x, t) = \sum_{i=0}^m \rho_i^m(t) w_i(x),$$

defined by the Galerkin ansatz, where

$$(2.35) \quad \begin{cases} \frac{\partial}{\partial t} v_n^m + (\Delta^2 - \Delta) v_n^m - (\Delta - Id) \left[\chi_n(\|v_n^m + \mathcal{L}(u_n)\|_q) P_m(f(v_n^m + \mathcal{L}(u_n))) \right] = 0, \\ v_n^m(x, 0) = P_m(u_0) \text{ in } \mathcal{D}, \quad \frac{\partial v_n^m}{\partial \nu} = \frac{\partial \Delta v_n^m}{\partial \nu} = 0 \text{ on } \partial \mathcal{D}. \end{cases}$$

This yields an initial value problem of ODE satisfied by $\rho_i^m(t)$ for $i = 0, 1, \dots, m$. By standard arguments of ODE, this initial value problem has a local solution. We will show that a global solution exists.

Multiplying by v_n^m both sides of (2.35), we obtain

$$(2.36) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|v_n^m(\cdot, t)\|_2^2 + \|\Delta v_n^m(\cdot, t)\|_2^2 + \|\nabla v_n^m(\cdot, t)\|_2^2 \\ &= \chi_n(\|v_n^m(\cdot, t) + \mathcal{L}(u_n)(\cdot, t)\|_q) \int_{\mathcal{D}} f(v_n^m(x, t) + \mathcal{L}(u_n)(x, t)) [\Delta v_n^m(x, t) - v_n^m(x, t)] dx \\ &= \sum_{i=1}^3 T_i(t), \end{aligned}$$

where

$$\begin{aligned} T_1(t) &= \chi_n(\|v_n^m(\cdot, t) + \mathcal{L}(u_n)(\cdot, t)\|_q) \int_{\mathcal{D}} [f(v_n^m(x, t) + \mathcal{L}(u_n)(x, t)) - f(v_n^m(x, t))] \\ &\quad \times [\Delta v_n^m(x, t) - v_n^m(x, t)] dx, \\ T_2(t) &= \chi_n(\|v_n^m(\cdot, t) + \mathcal{L}(u_n)(\cdot, t)\|_q) \int_{\mathcal{D}} f(v_n^m(x, t)) \Delta v_n^m(x, t) dx, \\ T_3(t) &= -\chi_n(\|v_n^m(\cdot, t) + \mathcal{L}(u_n)(\cdot, t)\|_q) \int_{\mathcal{D}} f(v_n^m(x, t)) v_n^m(x, t) dx. \end{aligned}$$

Since f is a polynomial of degree 3, then we have for $x, y \in \mathbb{R}$:

$$|f(x+y) - f(x)| \leq c|y|(1+x^2+y^2).$$

Thus by Cauchy-Schwarz and Young inequality we obtain for any $\varepsilon > 0$

$$\begin{aligned} T_1(t) &\leq C \chi_n(\|v_n^m(\cdot) + \mathcal{L}(u_n)(\cdot, t)\|_q) \int_{\mathcal{D}} |\mathcal{L}(u_n)(x, t)| [1 + |v_n^m(x, t)|^2 + |\mathcal{L}(u_n)(x, t)|^2] \\ &\quad \times [|\Delta v_n^m(x, t)| + |v_n^m(x, t)|] dx \\ &\leq C \chi_n(\|v_n^m(\cdot, t) + \mathcal{L}(u_n)(\cdot, t)\|_q) \|\mathcal{L}(u_n)(\cdot, t)\|_{\infty} [1 + \|v_n^m(\cdot, t)\|_2^2 + \|\mathcal{L}(u_n)(\cdot, t)\|_2^2] \\ &\quad \times [\|\Delta v_n^m(\cdot, t)\|_2 + \|v_n^m(\cdot, t)\|_2] \\ &\leq \varepsilon [\|\Delta v_n^m(\cdot, t)\|_2^2 + \|v_n^m(\cdot, t)\|_2^2] \\ &\quad + \frac{C}{\varepsilon} \chi_n(\|v_n^m(\cdot, t) + \mathcal{L}(u_n)(\cdot, t)\|_q) \|\mathcal{L}(u_n)(\cdot, t)\|_{\infty}^2 [1 + \|v_n^m(\cdot, t)\|_4^4 + \|\mathcal{L}(u_n)(\cdot, t)\|_4^4]. \end{aligned}$$

Observe that $f(x) = ax^3 + g(x)$, where $a > 0$ and g is a polynomial of degree 2. Hence, $f'(x) = 3ax^2 + 2bx + c$ for some real constants b, c , and $f'(x) \geq 2ax^2 - \tilde{c}$ for some non-negative

constant \tilde{c} . So, an integration by parts yields for any $\varepsilon > 0$

$$\begin{aligned}
T_2(t) &= -\chi_n(\|v_n^m(\cdot, t) + \mathcal{L}(u_n)(\cdot, t)\|_q) \int_{\mathcal{D}} f'(v_n^m(x, t)) |\nabla v_n^m(x, t)|^2 dx \\
&\leq C\chi_n(\|v_n^m(\cdot, t) + \mathcal{L}(u_n)(\cdot, t)\|_q) \int_{\mathcal{D}} [-2a|v_n^m(x, t)|^2 |\nabla v_n^m(x, t)|^2 + \tilde{c}|\nabla v_n^m(x, t)|^2] dx \\
&\leq -C\chi_n(\|v_n^m(\cdot, t) + \mathcal{L}(u_n)(\cdot, t)\|_q) \int_{\mathcal{D}} v_n^m(x, t) \Delta v_n^m(x, t) dx \\
&\leq \varepsilon \|\Delta v_n^m(\cdot, t)\|_2^2 + \frac{C}{\varepsilon} \chi_n(\|v_n^m(\cdot, t) + \mathcal{L}(u_n)(\cdot, t)\|_q) \|v_n^m(\cdot, t)\|_2^2.
\end{aligned}$$

Finally, since $xf(x) \geq \frac{7}{8}ax^4 - \tilde{C}$ with $a, \tilde{C} > 0$, we obtain

$$T_3(t) \leq \chi_n(\|v_n^m(\cdot, t) + \mathcal{L}(u_n)(\cdot, t)\|_q) \left[\int_{\mathcal{D}} -\frac{7}{8}a|v_n^m(x, t)|^4 dx + \tilde{C}|\mathcal{D}| \right].$$

The above upper estimates of $T_i(t)$, $i = 1, 2, 3$, imply that for $\varepsilon > 0$ small enough,

$$\begin{aligned}
(2.37) \quad &\frac{1}{2} \frac{d}{dt} \|v_n^m(\cdot, t)\|_2^2 + \frac{1}{2} \|\Delta v_n^m(\cdot, t)\|_2^2 + \|\nabla v_n^m(\cdot, t)\|_2^2 \leq C\chi_n(\|v_n^m(\cdot, t) + \mathcal{L}(u_n)(\cdot, t)\|_q) \\
&\times \left(\|\mathcal{L}(u_n(\cdot, t))\|_\infty^2 [1 + \|v_n^m(\cdot, t)\|_4^4 + \|\mathcal{L}(u_n(\cdot, t))\|_4^4] + \|v_n^m(\cdot, t)\|_2^2 + 1 \right).
\end{aligned}$$

Since

$$\|v_n^m(\cdot, 0)\|_2 = \|P_m u_0\|_2 \leq \|u_0\|_2,$$

integrating (2.37) from 0 to $t \in (0, T]$, and using Hölder's and Young's inequality, we obtain

$$\begin{aligned}
(2.38) \quad &\|v_n^m(\cdot, t)\|_2^2 + \int_0^t \|\Delta v_n^m(\cdot, s)\|_2^2 ds \leq \|u_0\|_2^2 + CT \left(1 + \|\mathcal{L}(u_n)\|_{L^\infty}^6 \right) \\
&+ C \left(1 + \|\mathcal{L}(u_n)\|_{L^\infty}^2 \right) \int_0^t \chi_n(\|v_n^m(\cdot, s) + \mathcal{L}(u_n)(\cdot, s)\|_q) (\|v_n^m(\cdot, s)\|_4^4 + 1) ds.
\end{aligned}$$

Since we have imposed $q \in [4, \infty)$, if the cut-off function χ_n applied to the $\|\cdot\|_q$ norm of some function U is not zero, we deduce that $\|U\|_4 \leq C(n+1) \leq Cn$. Recall that $|\chi_n| \leq 1$; thus the triangular inequality yields

$$\begin{aligned}
(2.39) \quad &\int_0^t \chi_n(\|v_n^m(\cdot, s) + \mathcal{L}(u_n)(\cdot, s)\|_q) (\|v_n^m(\cdot, s)\|_4^4 + 1) ds \leq CT(1 + \|\mathcal{L}(u_n)\|_{L^\infty}^4) \\
&+ C \int_0^t \chi_n(\|v_n^m(\cdot, s) + \mathcal{L}(u_n)(\cdot, s)\|_q) \|v_n^m(\cdot, s) + \mathcal{L}(u_n)(\cdot, s)\|_4^4 ds \\
&\leq CT(1 + \|\mathcal{L}(u_n)\|_{L^\infty}^4 + n^4).
\end{aligned}$$

The upper estimates (2.38) and (2.39) imply that

$$\begin{aligned}
(2.40) \quad &\sup_{t \in [0, T]} \|v_n^m(\cdot, t)\|_2^2 \leq \|u_0\|_2^2 + C(1 + \|\mathcal{L}(u_n)\|_{L^\infty}^6) + Cn^4 T(1 + \|\mathcal{L}(u_n)\|_{L^\infty}^4), \\
&\int_0^T [\|v_n^m(\cdot, t)\|_2^2 + \|\Delta v_n^m(\cdot, t)\|_2^2] dt \leq (T+1)\|u_0\|^2 + C(T^2+1)(1 + \|\mathcal{L}(u_n)\|_{L^\infty}^6) \\
&+ Cn^4(T^2+1)(1 + \|\mathcal{L}(u_n)\|_{L^\infty}^2).
\end{aligned}$$

Since the $H^2(\mathcal{D})$ -norm is equivalent to $\left(\int_{\mathcal{D}} (|\Delta u(x)|^2 + |u(x)|^2) dx \right)^{\frac{1}{2}}$ under the boundary condition $\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0$ on $\partial \mathcal{D}$, the right-hand side of (2.40) depends on n but is independent of the index m . Thus, a standard weak compactness argument proves that for fixed n , as $m \rightarrow \infty$, a subsequence of $(v_n^m, m \geq 1)$ converges weakly in $L^2(0, T; H^2(\mathcal{D}))$ to a solution v_n of (2.32) with homogeneous Neumann boundary conditions.

Let (ϵ_k) denote the orthonormal basis defined in Section 2.2 and set

$$(2.41) \quad B(u_0) := \frac{1}{2} \left\| \sum_{k \in \mathbb{N}^d} [\lambda_k + 1]^{-\frac{1}{2}} (u_0, \epsilon_k)_{L^2(\mathcal{D})} \epsilon_k \right\|_2^2 = \frac{1}{2} \sum_{k \in \mathbb{N}^d} [\lambda_k + 1]^{-1} (u_0, \epsilon_k)_{L^2(\mathcal{D})}^2.$$

Note that if \mathcal{D} is the unitary cube, we have $B(u_0) \leq \frac{1}{2} \|u_0\|_2^2$. The following lemma provides estimates of the $L^4(\mathcal{D})$ -norm of u_n .

Lemma 2.7. *Let σ be Lipschitz and satisfy the sub linearity condition (1.2) with $\alpha \in (0, 1]$, and let $u_0 \in L^q(\mathcal{D})$ where q satisfies Condition $(\tilde{\mathbf{C}}_\alpha)$. Let u_n be the solution to the SPDE (2.14) and $B(u_0)$ be defined by (2.41). Then there exists a constant $C := C(t, \mathcal{D})$ independent of the index n satisfying*

$$(2.42) \quad \int_0^t \chi_n(\|u_n(\cdot, s)\|_q) \|u_n(\cdot, s)\|_4^4 ds \leq C \left\{ 1 + B(u_0) + \int_0^t \chi_n(\|u_n(\cdot, s)\|_q) \|\mathcal{L}(u_n)(\cdot, s)\|_4^4 ds \right\}.$$

Proof. Using the orthonormal basis (ϵ_k) defined at the beginning of Section 2.2, we write $v_n \in L^2(\mathcal{D})$ as

$$v_n(x, t) = \sum_{k \in \mathbb{N}^d} \rho_k(t) \epsilon_k(x).$$

To ease notation, for $x \in \mathcal{D}$ and $s \in [0, T]$ we set

$$Q(x, s) := \chi_n(\|v_n(\cdot, s) + \mathcal{L}(u_n)(\cdot, s)\|_q) f(u_n(x, s)).$$

Then the equation (2.32) is written as follows

$$(2.43) \quad \partial_t v_n + (-\Delta + Id)(-\Delta)v_n + (-\Delta + Id)Q = 0,$$

with the boundary conditions $v_n(x, 0) = u_0(x)$ in \mathcal{D} , and $\frac{\partial v_n}{\partial \nu} = \frac{\partial \Delta v_n}{\partial \nu} = 0$ on $\partial\mathcal{D} \times [0, T]$.

We set $A = -\Delta + Id$, apply A^{-1} to (2.43) and take the L^2 inner product with $v_n(\cdot, t)$. The L^2 -orthogonality of the eigenfunctions ϵ_k of Δ gives

$$\sum_{k \in \mathbb{N}^d} [\lambda_k + 1]^{-1} \rho_k(t) \partial_t \rho_k(t) + \sum_{k \in \mathbb{N}^d} \lambda_k \rho_k(t)^2 + (Q(\cdot, t), v_n(\cdot, t)) = 0.$$

Integrating this identity from 0 to t yields

$$\int_0^t (Q(\cdot, s), v_n(\cdot, s)) ds = \sum_{k \in \mathbb{N}^d} \frac{1}{2} [\lambda_k + 1]^{-1} (\rho_k(0)^2 - \rho_k(t)^2) + \sum_{k \in \mathbb{N}^d} \left[[\lambda_k + 1]^{-1} (\rho_k(t)^2 - \rho_k(0)^2) + \int_0^t \lambda_k \rho_k(s)^2 ds \right].$$

Since $\lambda_k \geq 0$ for all k , we obtain

$$(2.44) \quad \int_0^t (Q(\cdot, s), v_n(\cdot, s))_{L^2} ds \leq \frac{1}{2} \sum_{k \in \mathbb{N}^d} [\lambda_k + 1]^{-1} \rho_k^2(0) = B(u_0).$$

Furthermore, f is a polynomial of degree 3; therefore, $f(u_n) \geq \frac{4}{5} u_n^4 - c$ for some non negative constant c . This yields

$$\int_0^t \chi_n(\|u_n(\cdot, s)\|_q) \|u_n(\cdot, s)\|_4^4 ds \leq C \left\{ 1 + \int_0^t \int_{\mathcal{D}} \chi_n(\|u_n(\cdot, s)\|_q) f(u_n(\cdot, s)) u_n(\cdot, s) dx ds \right\}.$$

Since $Q = \chi_n(\|v_n + \mathcal{L}(u_n)\|_q) f(u_n)$ and $u_n = \mathcal{L}(u_n) + v_n$, using (2.44) in the previous identity we obtain

$$(2.45) \quad \int_0^t \chi_n(\|u_n(\cdot, s)\|_q) \|u_n(\cdot, s)\|_4^4 ds \leq C \left\{ 1 + \int_0^t \int_{\mathcal{D}} \chi_n(\|u_n(\cdot, s)\|_q) f(u_n(\cdot, s)) \mathcal{L}(u_n(\cdot, s)) dx ds + B(u_0) \right\}.$$

Using once more the fact that $f(u_n)$ is a third degree polynomial, Young's inequality implies that for any $\epsilon > 0$ and $s \in [0, T]$,

$$\begin{aligned} \int_{\mathcal{D}} \chi_n(\|u_n(\cdot, s)\|_q) f(u_n(x, s)) \mathcal{L}(u_n(x, s)) dx &\leq \epsilon \int_{\mathcal{D}} \chi_n(\|u_n(\cdot, s)\|_q) |f(u_n(\cdot, s))|^{4/3} dx \\ &\quad + \frac{C}{\epsilon} \int_{\mathcal{D}} \chi_n(\|u_n(\cdot, s)\|_q) |\mathcal{L}(u_n(x, s))|^4 dx \\ &\leq C\epsilon \chi_n(\|u_n(\cdot, s)\|_q) \|u_n(\cdot, s)\|_4^4 + C + \frac{C}{\epsilon} \chi_n(\|u_n(\cdot, s)\|_q) \|\mathcal{L}(u_n(\cdot, s))\|_4^4. \end{aligned}$$

Consequently, plugging this upper estimate in (2.45) and choosing ϵ small enough, we complete the proof of (2.42). \square

2.7. Existence of a global solution. The Sobolev embedding theorem implies that for $d = 1, 2, 3$, $H^2(\mathcal{D}) \subset L^4(\mathcal{D})$. Hence, computations similar to that used to prove (2.38) with the weak $H^2(\mathcal{D})$ -limit v_n of v_n^m taken instead of v_n^m , show that for any $\epsilon > 0$ we have

$$\begin{aligned} \|v_n(\cdot, t)\|_2^2 + \int_0^t \|\Delta v_n(\cdot, s)\|_2^2 ds &\leq \|u_0\|_2^2 + C \int_0^t \tilde{T}_1(s) ds + \epsilon \int_0^t \|\Delta v_n(\cdot, s)\|_2^2 ds \\ &\quad + C(1 + \epsilon^{-1}) \left[T + \int_0^t \chi_n(\|v_n(\cdot, s) + \mathcal{L}(u_n)(\cdot, s)\|_q) \|v_n(\cdot, s)\|_4^4 ds \right], \end{aligned}$$

where the Cauchy-Schwarz and Young inequalities yield for any $\epsilon > 0$

$$\tilde{T}_1(s) \leq \epsilon [\|\Delta v_n(\cdot, s)\|_2^2 + \|v_n(\cdot, s)\|_2^2] + \frac{C}{\epsilon} \chi_n(\|v_n(\cdot, s) + \mathcal{L}(u_n)(\cdot, s)\|_q) \bar{T}_1(s),$$

for $\bar{T}_1(s)$ defined by

$$\begin{aligned} \bar{T}_1(s) &:= \int_{\mathcal{D}} |\mathcal{L}(u_n(x, s))|^2 [1 + |v_n(x, s)|^4 + |\mathcal{L}(u_n(x, s))|^4] dx \\ &\leq C [1 + \|\mathcal{L}(u_n(\cdot, s))\|_\infty^6 + \|\mathcal{L}(u_n(\cdot, s))\|_\infty^2 \|v_n(\cdot, s)\|_4^4]. \end{aligned}$$

Recall that $u_n = v_n + \mathcal{L}(u_n)$. Choosing ϵ small enough, using the Gronwall Lemma and Lemma 2.7, we deduce that for $t \in [0, T]$ there exists a positive constant $C(T)$ such that

$$\begin{aligned} \|v_n(\cdot, t)\|_2^2 + \frac{1}{2} \int_0^t \|\Delta v_n(\cdot, s)\|_2^2 ds &\leq C(T) \left(\|u_0\|_2^2 + 1 \right. \\ &\quad \left. + \int_0^t \chi_n(\|u_n(\cdot, s)\|_q) [1 + \|\mathcal{L}(u_n(\cdot, s))\|_\infty^6 + \|\mathcal{L}(u_n)\|_{L^\infty}^2 \|v_n(\cdot, s)\|_4^4] ds \right) \\ &\leq C(T) \left[\|u_0\|_2^2 + 1 + \|\mathcal{L}(u_n)\|_{L^\infty}^6 \right. \\ &\quad \left. + (1 + \|\mathcal{L}(u_n)\|_{L^\infty}^2) \int_0^t \chi_n(\|u_n(\cdot, s)\|_q) \|u_n(\cdot, s)\|_4^4 ds \right] \\ (2.46) \quad &\leq C(T) \left[1 + \|u_0\|_2^2 + \|\mathcal{L}(u_n)\|_{L^\infty}^6 + (1 + \|\mathcal{L}(u_n)\|_{L^\infty}^2) B(u_0) \right]. \end{aligned}$$

holds for every $n \geq 1$.

The following result is similar to (2.33) in [3], but the proof is different. Note that a gap in this reference has been fixed and that the proof is simpler using the Sobolev embedding Theorem and (2.28); it does not rely any more on an interpolation argument. As pointed out in [1], where stochastic existence for the Cahn-Hilliard equation has been proven for bounded domains of general geometry, the important property of the domain's boundary is being Lipschitz in dimensions 1, 2, 3. This fact together with the above H^2 -norm estimate (2.46) allows us to use easier L^∞ -norm arguments.

Lemma 2.8. *Let σ be Lipschitz and satisfy the sub linear growth condition (1.2) with $\alpha \in (0, 1]$. Let $u_0 \in L^q(\mathcal{D})$ where q satisfies Condition $(\tilde{\mathbf{C}}_\alpha)$, and let u_n be the solution to the SPDE (2.14). Then for any $\beta \in (1, \infty)$ and $a > 0$ such that $a\beta \in [2, \infty)$, there exists a positive constant $C(T)$ such that for every $n \geq 1$ we have*

$$(2.47) \quad \begin{aligned} E\left(\left[\int_0^T \|u_n(\cdot, t)\|_q^a dt\right]^\beta\right) &\leq C(T)[1 + \|u_0\|_2^{a\beta} + n^{3a\alpha\beta} + B(u_0)^{a\beta/2}(1 + n^{a\alpha\beta})] \\ &\leq C(T)[1 + \|u_0\|_2^{a\beta} + B(u_0)^{3a\beta/2} + n^{3a\alpha\beta}]. \end{aligned}$$

Proof. For any integer $k \geq 1$, set $\|u\|_{H^k(\mathcal{D})} := \left(\sum_{|a| \leq k} \|D_x^a u\|_{L^2(\mathcal{D})}^2\right)^{1/2}$. Using the Sobolev inequality (see e.g. [2], Theorem 1.4.6), we deduce the existence of a positive constant C such that for every $u \in H^k(\mathcal{D})$, $\|u\|_\infty \leq C\|u\|_{H^k(\mathcal{D})}$, provided that \mathcal{D} is a bounded domain with Lipschitz boundary and k is an integer with $k > d/2$. Therefore, if \mathcal{D} is a unit cube of \mathbb{R}^d , $d = 1, 2, 3$, we have

$$\|u\|_\infty \leq C\|u\|_{H^2(\mathcal{D})}.$$

Hence, given any $a \in (0, +\infty)$ and $\beta \in [1, +\infty)$ with $a\beta \geq 2$, since $u_n = v_n + \mathcal{L}(u_n)$, and \mathcal{D} is bounded and of Lipschitz boundary, then there exists a positive constant C depending on T , $|\mathcal{D}|$, a and β , such that for every integer $n \geq 1$:

$$\begin{aligned} \left(\int_0^T \|u_n(\cdot, t)\|_q^a dt\right)^\beta &\leq C \left[\sup_{t \in [0, T]} \|v_n(\cdot, t)\|_q^{a\beta} + \sup_{t \in [0, T]} \|\mathcal{L}(u_n)(\cdot, t)\|_q^{a\beta} \right] \\ &\leq C \left[\sup_{t \in [0, T]} \|v_n(\cdot, t)\|_\infty^{a\beta} + \|\mathcal{L}(u_n)\|_{L^\infty}^{a\beta} \right]. \end{aligned}$$

Thus, the inequalities (2.28) and (2.46) yield the existence of C as above such that

$$\begin{aligned} E\left[\left(\int_0^T \|u_n(\cdot, t)\|_q^a dt\right)^\beta\right] &\leq C[1 + \|u(0)\|_2^{a\beta} + B(u_0)^{a\beta/2}(1 + E\|\mathcal{L}(u_n)\|_{L^\infty}^{a\beta}) + E\|\mathcal{L}(u_n)\|_{L^\infty}^{3a\beta}] \\ &\quad + C(T)E\|\mathcal{L}(u_n)\|_{L^\infty}^{a\beta} \\ &\leq C[1 + \|u(0)\|_2^{a\beta} + B(u_0)^{a\beta/2} + B(u_0)^{a\beta/2}n^{a\alpha\beta} + n^{3a\alpha\beta}]. \end{aligned}$$

This proves the first upper estimate in (2.47). The second one is a straightforward consequence of the Young inequality applied with the conjugate exponents 3 and 3/2; this completes the proof. \square

The above lemma provides an upper estimate of moments of the q -norm of $\mathcal{M}(u_n)$.

Lemma 2.9. *Let σ be Lipschitz and satisfy the sub linearity condition (1.2) with $\alpha \in (0, 1]$, $u_0 \in L^q(\mathcal{D})$ where q satisfies Condition $(\tilde{\mathbf{C}}_\alpha)$. Let u_n be the solution of the SPDE (2.14) and let $\beta \in [\frac{2q}{q-d}, \infty)$. Then for $\mathcal{M}(u_n)$ defined by (2.16) there exists a positive constant $C := C(T)$ such that for and every $n \geq 1$ the following estimate holds:*

$$(2.48) \quad E\left(\sup_{0 \leq t \leq T} \|\mathcal{M}(u_n)(\cdot, t)\|_q^\beta\right) \leq C [1 + \|u_0\|_2^{3\beta} + B(u_0)^{9\beta/4} + n^{9\alpha\beta}].$$

Proof. Computations similar to that proving (2.19) yield for $\frac{1}{\rho} + \frac{1}{r} = \frac{1}{q} + 1$

$$\|\mathcal{M}(u_n)(\cdot, t)\|_q \leq C \int_0^t (t-s)^{-\frac{d+2}{4} + \frac{d}{4r}} \left\| \chi_n(\|u_n(\cdot, s)\|_q) f(u_n(y, s)) \right\|_\rho ds.$$

Since f is a third degree polynomial, choosing $\rho = \frac{q}{3}$ as before, we deduce

$$\left\| \chi_n(\|u_n(\cdot, s)\|_q) f(u_n(y, s)) \right\|_{\frac{q}{3}} \leq C \left(1 + \|u_n(\cdot, s)\|_q^3\right).$$

Thus for $\rho = \frac{q}{3}$ and r such that $\frac{2}{q} + \frac{1}{r} = 1$, we obtain for any $t \in [0, T]$:

$$\|\mathcal{M}(u_n)(\cdot, t)\|_q \leq C \int_0^t (t-s)^{-\frac{d+2}{4} + \frac{d}{4r}} \left(1 + \|u_n(\cdot, s)\|_q^3\right) ds.$$

Let $\gamma \in (1, \infty)$ be such that $(-\frac{d+2}{4} + \frac{d}{4r})\gamma > -1$, and γ' be the conjugate exponent. Then $\gamma' > \frac{2q}{q-d}$ and Hölder's inequality yields the existence of a positive constant C such that

$$\|\mathcal{M}(u_n)(\cdot, t)\|_q \leq C(T) \left\{ 1 + \left(\int_0^t \|u_n(\cdot, s)\|_q^{3\gamma'} ds \right)^{\frac{1}{\gamma'}} \right\}.$$

The upper estimate (2.47) completes the proof. Indeed, for $\beta > \frac{2q}{q-d}$ we can choose γ and γ' as above with $\frac{\beta}{\gamma'} > 1$; this clearly yields $3\beta \geq 2$. Then Hölder's inequality yields (2.48) for $\gamma'\beta \in [1, \infty)$. \square

In the above arguments; we only assumed that the exponent α appearing in the growth condition (1.2) was in the interval $(0, 1]$, including the case of a usual linear growth condition. However, to prove that equation (2.11) has a unique global solution, we must suppose that σ has a sub linear growth/ More precisely, we have to assume that $\alpha \in (0, 1/9)$.

For every integer $n \geq 1$ let us define the stopping time T_n as follows

$$(2.49) \quad T_n := \inf \left\{ t \geq 0 : \|u_n(\cdot, t)\|_q \geq n \right\}.$$

Then for every integer $n \geq 1$, the process $u(\cdot, t) = u_n(\cdot, t)$ is a solution of (2.11) on the interval $[0, T_n \wedge T]$. Assuming that $\alpha \in (0, \frac{1}{9})$, we will show that $\lim_{n \rightarrow \infty} T_n = \infty$ a.s., which will enable us to solve (2.11) on $[0, T]$ a.s. for any fixed T .

Theorem 2.10. *Suppose that σ is globally Lipschitz and satisfies the sub-linearity condition (1.2) with $\alpha \in (0, \frac{1}{9})$. Let $u_0 \in L^q(\mathcal{D})$ where q satisfies Condition $(\tilde{\mathbf{C}}_\alpha)$. Then for any $T > 0$ there exists a unique solution u to the SPDE (2.11) in the time interval $[0, T]$ (or equivalently if T_n is defined by (2.49), $T_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$); this solution belongs to $L^\infty([0, T]; L^q(\mathcal{D}))$ a.s. Furthermore, given any $\beta \in (\frac{2q}{q-d}, \infty)$, we have*

$$E \left(1_{\{T \leq T_n\}} \sup_{t \leq T} \|u(\cdot, t)\|_q^\beta \right) \leq C(1 + n^{9\alpha\beta}).$$

Proof. The sequence T_n is clearly non decreasing. Fix $T > 0$; by the definition of T_n , on the set $\{T_n < T\}$ we have for any $\beta \in [1, \infty)$

$$\sup_{t \in [0, T]} \|u_n(\cdot, t)\|_q^{2\beta} \geq n^{2\beta}.$$

Thus, the Chebyshev inequality, (2.12), (2.18), (2.28) and (2.48) yield the existence of a constant C depending on T , $\|u_0\|_q$ and $B(u_0)$ such that for every $n \geq 1$ the next inequality holds true

$$(2.50) \quad P(T_n < T) \leq n^{-2\beta} E \left(\sup_{t \in [0, T]} \|u_n(\cdot, t)\|_q^{2\beta} \right) \leq C n^{-2\beta(1-9\alpha)}.$$

Since β can be chosen large enough to ensure that $2\beta(1-9\alpha) > 1$, the Borel-Cantelli Lemma implies that $P(\limsup_n \{T_n < T\}) = 0$, that is $\lim_n T_n \geq T$ a.s. Since T is arbitrary, this yields $T_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$. The uniqueness of the solution to (2.14) implies that a process u can be uniquely defined setting $u(\cdot, t) = u_n(\cdot, t)$ on $[0, T_n]$. Since $T_n \rightarrow \infty$ a.s., we conclude that for any fixed $T > 0$, equation (2.11) has a unique solution and the upper estimate of moments of the q -norm of the solution follows from (2.12), (2.18), (2.28) and (2.48). \square

3. GENERALIZATION

The stochastic existence proof for the Cahn-Hilliard/Allen-Cahn equation with noise could easily be modified to hold for domains with more general geometry, cf. in [1] the proposed eigenvalue-formulae-free approach for the stochastic Cahn-Hilliard equation.

Our global existence and uniqueness result proven in Theorem 2.10 is also valid for the more general model

$$(3.1) \quad \begin{cases} u_t &= -\varrho \Delta (\Delta u - f(u)) + \tilde{q} (\Delta u - f(u)) + \sigma(u) \dot{W} & \text{in } \mathcal{D} \times [0, T), \\ u(x, 0) &= u_0(x) & \text{in } \mathcal{D}, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial \mathcal{D} \times [0, T), \end{cases}$$

for some constants $\varrho > 0$ and $\tilde{q} \geq 0$. The proof is very similar with the following simple modifications:

- (1) We have to replace the Green's function G defined by (2.2) by the following ϱ, \tilde{q} -dependent one

$$G^{\varrho, \tilde{q}}(x, y, t) := \sum_{k \in \mathbb{N}^d} e^{(-\varrho \lambda_k^2 + \tilde{q} \lambda_k)t} \epsilon_k(x) \epsilon_k(y).$$

All the estimates used on G also hold for $G^{\varrho, \tilde{q}}$, since the operator $-\varrho \Delta^2 + \tilde{q} \Delta$ is parabolic in the sense of Petrovskiĭ.

- (2) The estimate (2.42) also holds for $\varrho > 0$ and $\tilde{q} > 0$ when $B(u_0)$ is defined as

$$B(u_0) := \frac{1}{2} \left\| \sum_{k \in \mathbb{N}^d} [\varrho \lambda_k + \tilde{q}]^{-\frac{1}{2}} (u_0, \epsilon_k)_{L^2(\mathcal{D})} \epsilon_k \right\|_2^2.$$

Since $\varrho > 0$ and $\lambda_k \geq 0$, one may also invert $\varrho \lambda_k + \tilde{q}$ if $\tilde{q} > 0$ for any $k \in \mathbb{N}^d$.

If $\tilde{q} = 0$ (for $\varrho = 1$ we get the Cahn-Hilliard equation) then

$$B(u_0) := \frac{1}{2} \left\| \sum_{k \in \mathbb{N}^{*d}} [\varrho \lambda_k]^{-\frac{1}{2}} (u_0, \epsilon_k)_{L^2(\mathcal{D})} \epsilon_k \right\|_2^2.$$

and $\varrho \lambda_k$, for any $k \in \mathbb{N}^{*d}$, is invertible.

While the stochastic Cahn-Hilliard equation is a special case for our analysis (with $\varrho = 1$ and $\tilde{q} = 0$), this is not true for the stochastic Allen-Cahn equation. In our model the assumption that $\varrho > 0$ is crucial; indeed, since the fourth order operator is still acting, the operator $-\varrho \Delta^2 + \tilde{q} \Delta$ is also parabolic in the sense of Petrovskiĭ. Thus, the higher order differential operator is dominating and all the upper estimates of the Green's function and their derivatives stated in Section 2.2 remain valid.

4. PATH REGULARITY

In this section, we investigate the path regularity for the stochastic solution of (1.1) under certain regularity assumptions for the initial condition u_0 and when the domain \mathcal{D} is a parallelepiped. Note that the path regularity results proved in this section remain valid on a rectangular domain for the equation (3.1). More precisely, we prove that when the coefficient σ has an appropriate sub-linear growth, the paths of the solution to equation (1.1) have a.s. a Hölder regularity depending on that of the initial condition. The path regularity proven here is the same as that obtained for the stochastic Cahn-Hilliard equation obtained in [3], where the coefficient σ was supposed to be bounded. We follow the main lines of the proof presented in [3]; nevertheless some modifications are needed. Indeed, the factorization method is used both for the deterministic and stochastic integrals.

In this section we suppose that the assumptions of Theorem 2.10 are satisfied. Let us recall that the integral form of the solution u given by (2.11) can be decomposed as follows:

$$(4.1) \quad u(t, x) = G_t u_0(x) + \mathcal{I}(x, t) + \mathcal{L}(x, t),$$

where $G_t u_0$ is defined by (2.13), and

$$(4.2) \quad \begin{aligned} \mathcal{I}(x, t) &= \int_0^t \int_{\mathcal{D}} [\Delta G(x, y, t-s) - G(x, y, t-s)] f(u(y, s)) dy ds, \\ \mathcal{L}(x, t) &= \int_0^t \int_{\mathcal{D}} G(x, y, t-s) \sigma(u(y, s)) W(dy, ds). \end{aligned}$$

Let us study the regularity of each term in the decomposition (4.1) of u .

The series decomposition of G given in (2.2) is similar to that in [3]; hence an argument similar to the proof of Lemma 2.1 of [3] if u_0 is continuous, and to the first part of Lemma 2.2 of [3] if u_0 is δ -Hölder continuous, yields the following regularity result for $G.u_0(\cdot)$.

Lemma 4.1. *If u_0 is continuous, then the function $G_t u_0$ is continuous. If u_0 belongs to $C^\delta(\mathcal{D})$ for $0 < \delta < 1$, then the function $(x, t) \rightarrow G_t u_0(x)$ is δ -Hölder continuous in the space variable x and $\frac{\delta}{4}$ -Hölder continuous in the time variable t .*

Let us now consider the drift term $\mathcal{I}(x, t)$ and use the factorization method (see e.g. [8] or [3]).

We remark that, as proved in Theorem 2.10, if u_0 is bounded, then u belongs a.s. to $L^\infty(0, T; L^q(\mathcal{D}))$ for any $q < \infty$ large enough.

The definition of the Green's function yields

$$(4.3) \quad \Delta G(x, y, t) = \int_{\mathcal{D}} G(x, y, t-s) \Delta G(z, y, s) dz,$$

and

$$(4.4) \quad G(x, y, t) = \int_{\mathcal{D}} G(x, y, t-s) G(z, y, s) dz.$$

For some $a \in (0, 1)$ define the operators \mathcal{F} and \mathcal{H} on $L^\infty(0, T; L^q(\mathcal{D}))$ as follows:

$$\begin{aligned} \mathcal{F}(v)(t, x) &:= \int_0^t \int_{\mathcal{D}} G(x, z, t-s) (t-s)^{-a} v(z, s) dz ds, \\ \mathcal{H}(v)(z, s) &:= \int_0^s \int_{\mathcal{D}} [\Delta G(z, y, s-s') - G(z, y, s-s')] (s-s')^{a-1} f(v(y, s')) dy ds'. \end{aligned}$$

Therefore, using relations (4.3) and (4.4) we deduce that

$$\mathcal{I}(x, t) = c_a \mathcal{F}(\mathcal{H}(u))(x, t),$$

where $c_a := \pi^{-1} \sin(\pi a)$ obviously depends only on a .

First we claim that, for q satisfying condition $(\tilde{\mathbf{C}}_\alpha)$, the operator \mathcal{H} maps $L^\infty(0, T; L^q(\mathcal{D}))$ into itself. Indeed, the estimates on the Green's function in Lemma 2.1 and arguments similar to those used in Section 2.4 to prove (2.19) with $\rho = \frac{q}{3}$ (based on the Minkowski and Young inequalities) prove that if $v \in L^\infty(0, T; L^q(\mathcal{D}))$ then

$$\|\mathcal{H}(v)(\cdot, t)\|_q \leq \int_0^t (t-s)^{-1+a-\frac{1}{2}-\frac{d}{2q}} (1 + \|v(\cdot, s)\|_q^3) ds.$$

For the boundedness of the above integral we need that $-1+a-\frac{1}{2}-\frac{d}{2q} > -1$. Since $q > d$, this inequality holds for some $a \in (\frac{1}{2} + \frac{d}{2q}, 1)$. Then, an argument similar to that used in [3] proves that if $v \in L^\infty([0, T]; L^q(\mathcal{D}))$ then $\mathcal{J}(v)$ belongs to $\mathcal{C}^{\lambda, \mu}(\mathcal{D} \times [0, T])$ for any $\lambda < 1$ and $\mu < \frac{1}{2}$. Indeed, the upper estimates of the Green's function from Lemma 2.2 are the same as

that for the Green's function of the Cahn-Hilliard equation which only involves the fourth order derivatives.

Considering the stochastic integral \mathcal{L} defined in (4.2), we observe that the fact that σ is not bounded any more does not allow us to use the related argument from the proof of Lemma 2.2 in [3] stated on page 797. Instead, we also use the factorization method for the stochastic integral. Recall that given any $n \geq 1$, for $T_n = \inf\{t \geq 0 : \|u_n(\cdot, t)\|_q \geq n\}$ we have $1_{\{T \leq T_n\}}u(\cdot, t) = 1_{\{T \leq T_n\}}u_n(\cdot, t)$, where u_n is the solution to (2.14). The local property of stochastic integrals implies that for any n and $t \in [0, T]$:

$$1_{\{T \leq T_n\}}\mathcal{L}(x, t) = 1_{\{T \leq T_n\}} \int_0^t \int_{\mathcal{D}} G(t-s, x, y) 1_{\{s \leq T_n\}} \sigma(u_n(y, s)) W(dy, ds).$$

The process u_n is adapted and by (2.47) for q large enough, if Condition (C) holds true, $\gamma > 0$ and $\beta \in (1, \infty)$ are such that $\beta\gamma \in [2, \infty)$, we have $E \left| \int_0^T \|u_n(\cdot, t)\|_q^\gamma dt \right|^\beta \leq C(n, T)$. Fix $a \in (0, 1)$, let $\mathcal{K}(u_n)$ be defined as follows:

$$\mathcal{K}(u_n)(x, t) = \int_0^t \int_{\mathcal{D}} G(x, y, t-s) (t-s)^{a-1} 1_{\{s \leq T_n\}} \sigma(u_n(y, s)) W(dy, ds).$$

We at first check that this stochastic integral makes sense for fixed $t \in [0, T]$ and $x \in \mathcal{D}$, and that a.s. $\mathcal{K}(u_n) \in L^\infty(0, T; L^q(\mathcal{D}))$, so that $1_{\{T \leq T_n\}}\mathcal{L}(x, t) = 1_{\{T \leq T_n\}}c_a \mathcal{F}(\mathcal{K}(u_n))(x, t)$.

Indeed, for fixed $t \in [0, T]$, $x \in \mathcal{D}$ and $p \in [1, \infty)$, the Burkholder inequality yields

$$\begin{aligned} E|\mathcal{K}(u_n)(x, t)|^{2p} &\leq E \left| \int_0^t \int_{\mathcal{D}} G^2(x, y, t-s) (t-s)^{2(a-1)} 1_{\{s \leq T_n\}} \sigma^2(u_n(s, y)) dy ds \right|^p \\ &\leq C(n) \left| \int_0^t (t-s)^{-\frac{d}{2}+2(a-1)+\frac{d}{4}} ds \right|^p \end{aligned}$$

Let $a \in (\frac{1}{2} + \frac{d}{8}, 1)$; then we have $-\frac{d}{2} + 2(a-1) + \frac{d}{4} > -1$, which yields

$$(4.5) \quad E|\mathcal{K}(u_n)(x, t)|^{2p} < \infty, \quad \forall p \in [1, \infty).$$

Let us now prove moment upper estimates of increments of $\mathcal{K}(u_n)$; this together with (4.5) will imply by Garsia's Lemma that

$$E(\|\mathcal{K}(u_n)\|_{L^\infty(\mathcal{D} \times [0, T])}^{2\rho}) < \infty.$$

Arguments similar to those used in the proof of (2.30) prove that for $\tilde{\lambda} \in (0, 1)$, $\tilde{q} \in (2\alpha, q)$ and $n \geq 1$, we have for $t \in [0, T]$, $x, \xi \in \mathcal{D}$:

$$\begin{aligned} E|\mathcal{K}(u_n)(x, t) - \mathcal{K}(u_n)(\xi, t)|^{2p} &\leq C_p |x - \xi|^{2\tilde{\lambda}p} \\ &\quad \times \left| \int_0^t (t-s)^{-\frac{d+1}{2}\tilde{\lambda} - \frac{d}{2}(1-\tilde{\lambda}) + 2(a-1)} 1_{\{s \leq T_n\}} \|\exp(h(\cdot, t, s))\|_{\frac{\tilde{q}}{\tilde{q}-2\alpha}} [1 + \|u_n(\cdot, s)\|_{\tilde{q}}^{2\alpha}] ds \right|^p \\ &\leq C_p(n) |x - \xi|^{2\tilde{\lambda}p} \left| \int_0^t (t-s)^{-\frac{d+\tilde{\lambda}}{2} + 2(a-1) + \frac{d(\tilde{q}-2\alpha)}{4\tilde{q}}} ds \right|^p \\ &\leq C_p(n, T) |x - \xi|^{2\tilde{\lambda}p} \end{aligned}$$

for some finite constant $C_p(n, T)$, provided that the time integrability constraint $-\frac{d+\tilde{\lambda}}{2} + 2(a-1) + \frac{d(\tilde{q}-2\alpha)}{4\tilde{q}} > -1$ holds true. Since Condition $(\tilde{\mathbf{C}}_\alpha)$ is satisfied with $\alpha \in (0, \frac{1}{9})$, we deduce that $q > \frac{2\alpha d}{4-d}$. Hence given $\tilde{\lambda} \in (0, 2 - \frac{d}{2})$ one can find $\tilde{q} \in (\frac{2\alpha d}{4-d}, q)$ and $a \in (\frac{1}{2} + \frac{d}{8}, 1)$ such that the time integrability is fulfilled.

Similarly, for $0 \leq t' \leq t \leq T$, $x \in \mathcal{D}$ and $\tilde{\mu} \in (0, \frac{1}{2} - \frac{d}{8})$, arguments similar to that proving (2.31) imply

$$\begin{aligned} E|\mathcal{K}(u_n)(x, t) - \mathcal{K}(u_n)(x, t')|^{2p} &\leq C_p |t - t'|^{2\tilde{\mu}p} \\ &\times \left| \int_0^t (t-s)^{-2(\frac{d}{4}+1)\tilde{\mu} - \frac{d}{2}(1-\tilde{\mu})+2(a-1)} 1_{\{s \leq T_n\}} \|\exp(h(\cdot, t, s))\|_{\frac{\tilde{q}}{\tilde{q}-2\alpha}} [1 + \|u_n(\cdot, s)\|_{\tilde{q}}^{2\alpha}] ds \right|^p \\ &\leq C_p(n) |t - t'|^{2\tilde{\mu}p} \left| \int_0^t (t-s)^{-\frac{d}{2}-2\tilde{\mu}+2(a-1)+\frac{d(\tilde{q}-2\alpha)}{4\tilde{q}}} ds \right|^p \\ &\leq C_p(n, T) |t - t'|^{2\tilde{\mu}p} \end{aligned}$$

for some finite constant $C_p(n, T)$, provided that $2a - 2\tilde{\mu} > 1 + \frac{d}{4} + \frac{\alpha d}{2\tilde{q}}$. Once more, since $\alpha \in (0, \frac{1}{9})$, we have $q > \frac{2\alpha d}{4-d}$ and given $\tilde{\mu} \in (0, \frac{1}{2} - \frac{d}{8})$, we can find $\tilde{q} \in (\frac{2\alpha d}{4-d}, q)$ such that this inequality holds true.

Hence, given $\bar{\lambda} \in (0, 2 - \frac{d}{2})$ and $\bar{\mu} \in (0, \frac{1}{2} - \frac{d}{8})$, for every $n \geq 1$ and $p \in [1, \infty)$, we can find some positive constant $C_p(n, T)$ such that

$$E|\mathcal{K}(u_n)(x, t) - \mathcal{K}(u_n)(\xi, t')|^{2p} \leq C_p(n, T) (|\xi - x|^{2\bar{\lambda}p} + |t - t'|^{2\bar{\mu}p})$$

for $0 \leq t' \leq t \leq T$ and $x, \xi \in \mathcal{D}$.

The Garsia-Rodemich-Rumsey lemma implies that

$$E(\|\mathcal{K}(u_n)\|_{L^\infty(\mathcal{D} \times [0, T])}^{2p}) < \infty, \quad \forall p \geq 1,$$

and

$$E(\|\mathcal{K}(u_n)\|_q^{2p}) \leq E(\|\mathcal{K}(u_n)\|_{L^\infty(\mathcal{D} \times [0, T])}^{2p}) < \infty, \quad \forall p \geq 1.$$

This gives on one hand the stated time and space Hölder regularity, and on the other hand the previous space-time Hölder moments estimates of $\mathcal{K}(u_n) \in L^\infty(0, T, L^q(\mathcal{D}))$ a.s.

Since \mathcal{F} maps $L^\infty(0, T; L^q(\mathcal{D}))$ into $\mathcal{C}^{\lambda, \mu}(\mathcal{D} \times [0, T])$ for $\lambda < \frac{1}{2} - \frac{d}{8}$ and $\mu < 2 - \frac{d}{2}$ and since $\mathcal{L}(x, t) = c_a \mathcal{F}(\mathcal{K}(u_n))(x, t)$ on the set $\{T \leq T_n\}$, we deduce that $\mathcal{L}(u) \in \mathcal{C}^{\lambda, \mu}(\mathcal{D} \times [0, T])$ a.s. on the set $\{T \leq T_n\}$.

Finally, Theorem 2.10 implies that as $n \rightarrow \infty$ the sets $\{T \leq T_n\}$ increase to Ω ; this proves that a.s. $\mathcal{L}(u) \in \mathcal{C}^{\lambda, \mu}([0, T] \times \mathcal{D})$ for $\lambda < \frac{1}{2} - \frac{d}{8}$ and $\mu < 2 - \frac{d}{2}$.

As a consequence (cf. [3]), we obtain the following regularity of the trajectories.

Theorem 4.2. *Let σ be Lipschitz and satisfy the sub linearity condition (1.2) with $\alpha \in (0, \frac{1}{9})$, let q satisfy Condition $(\tilde{\mathcal{C}}_\alpha)$, and let $u_0 \in L^q(\mathcal{D})$. Then*

- (i) *If u_0 is continuous, then the solution of (2.11) has almost surely continuous trajectories.*
- (ii) *If u_0 is β -Hölder continuous for $0 < \beta < 1$, then the trajectories of the solution to (2.11) are almost surely $\beta \wedge (2 - \frac{d}{2})$ -continuous in space and $\frac{\beta}{4} \wedge (\frac{1}{2} - \frac{d}{8})$ -continuous in time.*

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